Analytical results for coupled-map lattices with long-range interactions

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We obtain exact analytical results for lattices of maps with couplings that decay with distance as $r^{-\alpha}$. We analyze the effect of the coupling range on the system dynamics through the Lyapunov spectrum. For lattices whose elements are piecewise linear maps, we get an algebraic expression for the Lyapunov spectrum. When the local dynamics is given by a nonlinear map, the Lyapunov spectrum for a completely synchronized state is analytically obtained. The critical line characterizing the synchronization transition is determined from the expression for the largest transversal Lyapunov exponent. In particular, it is shown that in the thermodynamical limit, such transition is only possible for sufficiently long-range interactions, namely, for $\alpha < \alpha_c = d$, where d is the lattice dimension.

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Synchronization between coupled chaotic systems is one of the most intriguing nonlinear phenomena [1]. It has been attracting much interest since the past two decades [2], as it appears in a wide range of real systems such as in arrays of Josephson junctions [3], oscillating chemical reactions [4], physiological processes [5], and has applications as in communications [6] and control theory [7]. There are many types of synchronized behavior [8], but we are particularly interested in the completely synchronized states (CSSs) of coupled-map lattices (CMLs), where all maps present the same amplitude at all times. Complete synchronization is an example of nonequilibrium phase transition [9], which may be related to actual critical phenomena such as the superconducting-normal transition in Josephson junctions [10].

CMLs, which are dynamical systems with discrete space and time, and a continuous state variable, have been investigated as theoretical models of spatiotemporal phenomena in a variety of problems in condensed matter physics, neuroscience, and chemical physics [11]. The spatiotemporal behavior is governed by two simultaneous mechanisms: the intrinsic nonlinear dynamics of each map, and diffusion due to the spatial coupling between maps; the dynamical pattern being the outcome of the competition between them. This applies, in particular, to the problem of synchronization of chaotic maps [12]. The effective coupling range is a crucial factor to determine whether or not chaotic maps mutually synchronize. Nearest-neighbor couplings (short range) do not favor synchronization, since the coupling effect is typically too weak to overcome the intrinsic randomness of map dynamics [13]. On the other hand, long-range couplings tend to facilitate synchronization, as exemplified by the limiting case of global (mean-field) coupling [14]. Lattices of nonlocally coupled maps appear in neural networks with local production of information [15], models of physicochemical reactions [16], assemblies of biological cells with oscillatory activity [17], and diffusion coupling in nucleation kinetics [18]. Beyond CMLs, systems with many degrees of freedom with long-range couplings are an interesting object of study because of their anomalies (appearing at the level of the macroscopic thermodynamical description as well as in the un-

derlying microscopic dynamics), which still require deeper understanding [19]. Simple dynamical models, such as CMLs, may add new knowledge on nonequilibrium long-range systems. However, there is a lack of analytical results for CMLs with arbitrary range couplings. Exact analytical results are particularly crucial because the occurrence of phenomena such as shadowing breakdown [20] or spurious synchronization [21] set difficulties in numerical approaches due to the unavoidable finite precision of numerical simulations.

Here we examine a form of coupling whose intensity decays with the distance r between sites as $1/r^\alpha$, with $\alpha \geq 0$ [22]. It has also been considered in biological networks [23], in ferromagnetic spin models [24], many-particle conservative (Hamiltonian time evolution) classical systems [25,26], large populations of limit cycle oscillators [27], and a generalization of the Kuramoto model [28], among other examples. Explicitly, we consider a chain of N coupled one-dimensional chaotic maps $x \mapsto f(x)$ such that the coupling prescription is

$$x_{n+1}^{(i)} = (1-\varepsilon)f(x_n^{(i)}) + \frac{\varepsilon}{\eta(\alpha)} \sum_{r=1}^{N'} \frac{f(x_n^{(i-r)}) + f(x_n^{(i+r)})}{r^\alpha},$$

(1)

where $x_n^{(i)}$ represents the state variable for the site i ($i = 1,2,\ldots,N$) at time n, $\varepsilon \geq 0$ and $\alpha \geq 0$ are the coupling strength and effective range, respectively, and $\eta(\alpha) = 2\sum_{r=1}^{N'} r^{-\alpha}$ is the normalization factor, with $N' = (N-1)/2$ for odd N. In conservative systems [25,26], scaling by η plays an important role in making the systems pseudoextensive. Here periodic boundary conditions $x_n^{(i)} = x_n^{(i \pm N)}$ and random initial conditions are assumed. The coupling term is a weighted average of discretized spatial second derivatives, the normalization factors being the sum of the corresponding statistical weights. It is straightforward to prove that in the limits $\alpha = 0$ and $\alpha \to \infty$, Eq. (1) reduces to the global mean-field and the local Laplacian-type couplings, respectively.

We characterize the spatiotemporal synchronization dynamics by means of the Lyapunov spectrum (LS) of the lattice, which enables one to estimate, for instance, the Kolmogorov-Sinai (KS) entropy through the Pesin formula

[29] and the Lyapunov dimension, which gives an upper bound on the effective number of degrees of freedom needed to characterize the system dynamics [30]. Besides characterizing a CSS, when it exists at all, we must investigate its stability with respect to small perturbations. If the CSS turns out to be dynamically unstable, we are faced with two possibilities: either the CSS presents the so-called bubbling attractor, and in this case the CSS only lasts for a finite time, or the CSS loses transversal stability through a blowout bifurcation [8].

In this work we will present exact analytical results for the CML (1). We will show that for a one-dimensional (1D) lattice of N coupled piecewise linear maps it is possible to obtain an exact analytical expression for the LS, the results shown in Refs. [14,29] being recovered in the limits $\alpha=0$ and $\alpha \to \infty$. When the maps $x \mapsto f(x)$ are nonlinear, we will show that analytical results are still possible for CSSs. By means of the algebraic formulas for the LS, one can find the synchronization regions in the $\varepsilon \times \alpha$ space, since the second largest Lyapunov exponent [belonging to the direction transversal to the synchronization manifold (SM)] equal to zero indicates a transition to the synchronized state [16,31]. Finally, the results obtained for a chain of maps will be extended to d-dimensional hypercubic lattices.

In order to calculate the LS one has to consider the tangent dynamics. By differentiating the equations of the original maps (1), one obtains the evolution equations for tangent vectors $\xi = (\delta x^{(1)}, \delta x^{(2)}, \ldots, \delta x^{(N)})^T$, which in matrix form read $\xi_{n+1} = \mathbf{T}_n \xi_n$, with the Jacobian matrix $\mathbf{T}_n$ given by

$$\mathbf{T}_n = \left[(1-\varepsilon) + \frac{\varepsilon}{\eta(\alpha)} \mathbf{B} \right] \mathbf{D}_n, \qquad (2)$$

where the matrices $\mathbf{D}_n$ and $\mathbf{B}$ are defined, respectively, by $D_n^{jk} = f'(x_n^{(j)}) \delta_{jk}$ and $B_{jk} = 1/r_{jk}^{\alpha}(1-\delta_{jk})$, where $r_{jk} = \min_{l \in \mathbb{Z}} |j-k+lN|$. Notice that the particular choice of the interaction law is embodied in the matrix $\mathbf{B}$ which is time independent.

Once specified the initial conditions, the LS is extracted from the evolution of the initial tangent vector ξ_0: $\xi_n = \mathcal{T}_n \xi_0$, where $\mathcal{T}_n \equiv \mathbf{T}_n \mathbf{T}_{n-1} \cdots \mathbf{T}_2 \mathbf{T}_1$ is product of n Jacobian matrices calculated at successive points of the discrete trajectory. If $\Lambda_1, \ldots, \Lambda_N$ are the eigenvalues of $\hat{\Lambda} = \lim_{n \to \infty} (\mathcal{T}_n \mathcal{T}_n^T)^{1/2n}$ (that are real and positive), the Lyapunov exponents are obtained as [32]

$$\lambda_k = \ln \Lambda_k, \qquad k=1, \ldots, N. \qquad (3)$$

We start by applying the expressions above to the piecewise linear maps $x \mapsto f(x) = \beta x \pmod{1}$, with $\beta \geq 1$. In this case we have $f'(x) = \beta = \text{const}$, therefore $\mathbf{D}_n = \beta \mathbb{1}_N$, and $\mathbf{T}_n$ becomes

$$\mathbf{T}_n = \beta \left[(1-\varepsilon) \mathbb{1}_N + \frac{\varepsilon}{\eta(\alpha)} \mathbf{B} \right] \equiv \beta \hat{\mathbf{B}}, \qquad (4)$$

the rightmost identity defining the matrix $\hat{\mathbf{B}}$. Since the symmetric tangent map does not depend on time, it results $\hat{\Lambda}$

$= \beta \hat{\mathbf{B}}$. So, in order to obtain the LS, it is enough to diagonalize $\mathbf{B}$. Because of its periodicity, $\mathbf{B}$ can be diagonalized in Fourier space [33], the eigenvalues being

$$b_k = 2 \sum_{m=1}^{N'} \frac{\cos(2\pi km/N)}{m^\alpha}, \qquad k=1, \ldots, N, \qquad (5)$$

where we considered odd N. Finally, from Eq. (3), taking into account the special form of $\hat{\Lambda}$, the LS is given by

$$\lambda_k = \ln \beta + \ln \left| 1 - \varepsilon + \frac{\varepsilon}{\eta(\alpha)} b_k \right|. \qquad (6)$$

This expression is consistent with previous numerical results [34]. In the extreme cases $\alpha \to \infty$ and $\alpha=0$ the known expressions [14,29] are recovered.

Now we will consider lattices of nonlinear maps. An important case that can be tackled easily is the one where the maps are in the CSS. As it will become clear soon, this instance provides relevant information on the synchronization transition. In the CSS, the dynamical variables of all maps coincide, i.e., $x_n^{(1)} = x_n^{(2)} = \cdots = x_n^{(N)} \equiv x_n^{(*)}$, at each time step n. The LS for the CCS when $\alpha=0$ has already been found by Kaneko [14]. Now, for arbitrary α, we have $\mathbf{D}_n = f'(x_n^{(*)}) \mathbb{1}_N$, thus, $\mathbf{T}_n = f'(x_n^{(*)}) \hat{\mathbf{B}}$ and $\mathcal{T}_n \mathcal{T}_n^T = (\Pi_{j=1}^n [f'(x_j^{(*)})]^2) \hat{\mathbf{B}}^{2n}$. Therefore, for the CSS, following Eq. (3), one arrives at

$$\lambda_k^* = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \ln |f'(x_i^{(*)})| + \ln \left| 1 - \varepsilon + \frac{\varepsilon}{\eta(\alpha)} b_k \right|, \qquad (7)$$

where b_k are the eigenvalues of $\mathbf{B}$ defined in Eq. (5). Assuming ergodicity, the time average in Eq. (7) can be substituted by an average over the single-map attractor. In this way one gets

$$\lambda_k^* = \lambda_U + \ln \left| 1 - \varepsilon + \frac{\varepsilon}{\eta(\alpha)} b_k \right|, \qquad (8)$$

where $\lambda_U = \langle \ln |f'(x^{(*)})| \rangle$ is the Lyapunov exponent of an uncoupled map. This expression is *general*: it applies to any lattice of nonlinear 1D maps coupled with the scheme here considered, the parameters that define the particular uncoupled map affecting only λ_U. For instance, for the logistic map $x \mapsto f(x) = ax(1-x)$, with $a=4$ and $x \in [0,1]$, $\lambda_U = \langle \ln |4(1-2x^{(*)})| \rangle = \ln 2$ [35] and the contribution of the power-law coupling is always $\ln |1 - \varepsilon + [\varepsilon/\eta(\alpha)] b_k|$. Notice that the LS in CSSs has the same structure as the LS obtained for piecewise linear maps [Eqs. (5) and (6)].

The synchronization transition can be characterized by a complex order parameter [4] defined, for time n, as $R_n = |(1/N) \Sigma_{j=1}^N e^{2\pi i x_n^{(j)}}|$. A time-averaged amplitude $\bar{R}$ is computed over an interval large enough to warrant that the lattice has attained the asymptotic state. In the CSS, one has $\bar{R} = 1$. On the opposite case of completely nonsynchronized maps, the site state variables $x_n^{(j)}$ are so uncorrelated that $\bar{R} \sim O(N^{-1/2})$.
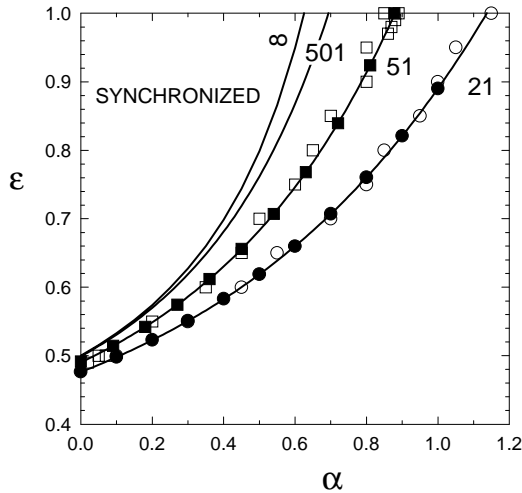
FIG. 1. Synchronization domains in parameter plane $\varepsilon \times \alpha$ for a 1D lattice of N coupled logistic maps $x \mapsto f(x) = 4x(1-x)$. The critical line $\varepsilon_c(\alpha)$ was determined analytically from Eq. (9) for different values of N (full lines) and numerically from the conditions $\bar{R} \neq 1$ (open symbols) and $\tilde{\lambda}_2^* = 0$ (full symbols) for $N = 21$ and 51.
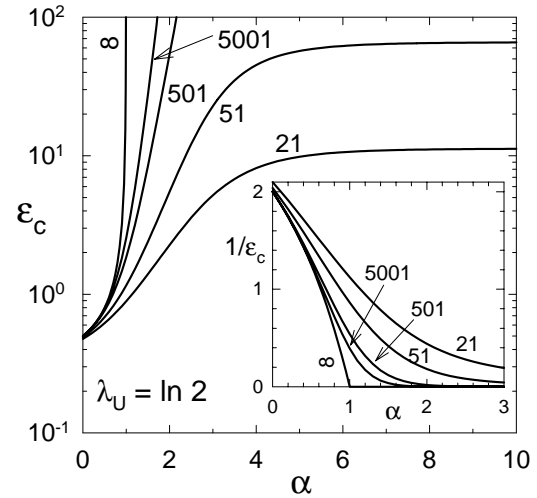

FIG. 2. Critical coupling strength for synchronization vs range parameter, for different values of N indicated in the figure. The critical lines were determined analytically from Eqs. (9) and (10) for a generic 1D lattice with power-law couplings and $\lambda_U = \ln 2$. Inset: the same data in a $1/\varepsilon_c$ vs α plot.

A diagnostic of synchronization can also be extracted from the LS. It can be easily verified that, for arbitrary α, the CSS lies along the direction of the eigenvector associated with the largest exponent. This was previously observed by Kaneko for the particular case $\alpha = 0$ [14]. Therefore, the CSS will be transversally stable if the $(N-1)$ remaining exponents are nonpositive, that is, $\tilde{\lambda}_2^* \leq 0$ (where the tilde stands for ordered exponents). The second largest exponent $\tilde{\lambda}_2^*$ corresponds in Eq. (8) to $k = 1$ (or $k = N-1$, due to degeneracy). The analytical expression for the critical line $\varepsilon_c(\alpha)$ results in

$$\varepsilon_c = (1 - e^{-\lambda_U}) \left(1 - \frac{2}{\eta(\alpha)} \sum_{m=1}^{N'} \frac{\cos(2\pi m/N)}{m^\alpha} \right)^{-1}, \quad (9)$$

where ε_c is the coupling strength below which the SM ceases to be transversely stable, such that synchronized states are not typically observed. It is noteworthy that Eq. (9) is *quite general* for the coupling scheme here considered: the parameters that define the particular uncoupled nonlinear map (embedded in λ_U) just participate through the first factor.

Figure 1 presents the synchronization regions in the parameter space $\varepsilon \times \alpha$ for N coupled logistic maps $x \mapsto f(x) = 4x(1-x)$. The critical line was obtained analytically from Eq. (9) and numerically by means of two procedures: either when $\bar{R} \neq 1$ is numerically detected or by the condition of nullity of the second largest Lyapunov exponent in the CSS. The numerical results are in good agreement with the analytical prediction.

As can be observed in Fig. 1, the critical frontier depends on the system size N. In the limit $N \to \infty$ we obtain

$$\varepsilon_{c,\infty} = \frac{1 - e^{-\lambda_U}}{1 - C(\alpha)}, \quad (10)$$

where $C(\alpha)$ corresponds to

$$C(\alpha) = \lim_{N \to \infty} \left(\left[\sum_{m=1}^{N'} \frac{\cos(2\pi m/N)}{m^\alpha} \right] \left[\sum_{m=1}^{N'} \frac{1}{m^\alpha} \right]^{-1} \right). \quad (11)$$

This limit is equal to unity for $\alpha > 1$, so that Eq. (10) furnishes a divergent result. For α outside the domain of convergence of the series, one gets

$$C(\alpha) = \frac{1-\alpha}{\pi^{1-\alpha}} \int_0^\pi \frac{\cos(x)}{x^\alpha} dx. \quad (12)$$

So, if the system is finite, synchronization can be achieved for any α provided the coupling strength ε is large enough. However, in the limit $N \to \infty$, synchronization is only possible for sufficiently long-range interactions (see Fig. 2), namely, for $\alpha < \alpha_c = 1$. This critical value here obtained for 1D CMLs is different from the one reported for other 1D systems with similar power-law interactions, such as ferromagnetic spin models [24] or many-particle classical Hamiltonian systems [25]; in such cases, the critical value for the existence of an order/disorder transition is $\alpha_c = 2$. In a generalization of the Kuramoto model, the value of α_c is controversial: While $\alpha_c = 2$ was first reported [28], recent analytical considerations point to $\alpha_c = 1$ [36]. Although the generalized Kuramoto model is a continuous-time dynamical system, our analytical result (namely, $\alpha_c = 1$) suggests that the latter value is the correct one.

The expressions we have derived for the LS of chains of maps can be straightforwardly generalized for hypercubic lattices of arbitrary dimension d. In fact, for the general

d-dimensional case, it is straightforward to show that the eigenvalues of the matrix **B** become

$$b_k = 2^d \sum_{\bar{m} \neq 0} \frac{\cos(2\pi\bar{r}_k \cdot \bar{m}/N^{1/d})}{m^\alpha}, \quad k = 1, \ldots, N, \quad (13)$$

where $\bar{r}_k$ is the position vector of site k, $\bar{m} = (m_1, \ldots, m_d)$, with $0 \leq m_i \leq N'$, and $N' = (N^{1/d} - 1)/2$. The normalization factor reads

$$\eta(\alpha, d) = 2^d \sum_{\bar{m} \neq 0} \frac{1}{m^\alpha} \propto \frac{N^{\alpha/d} - 1}{1 - \alpha/d}. \quad (14)$$

Then, following the same lines as before but now for the sake of generalizing Eq. (10), it is easy to see that $\varepsilon_{c,\infty}$ will diverge if $\alpha/d > 1$, which leads to $\alpha_c = d$.

Additionally, our results for the LS could be extended to the more general class of coupling schemes where the dependence of the coupling strength on the intermap distance is not necessarily of the power-law type. In these cases, one should feed Eqs. (6) and (8) with the eigenvalues of the appropriate matrix **B**, which contains the particular dependence of the interaction strength on distance.

In conclusion, we have presented analytical expressions for the LS of CMLs with an interaction which decays with the lattice distance as a power law, for two cases: (i) piecewise linear coupled maps, and (ii) the CSS of lattices of one-dimensional maps. Our results enable us to predict the critical values for synchronization in the coupling parameter plane (strength versus range), and also may be used to obtain related quantities of interest, such as KS entropies and Lyapunov dimensions. Such exact analytical results are crucial in order to avoid difficulties present in numerical approaches, such as shadowing breakdown, due to unavoidable finite precision of numerical simulations. In addition, we have shown that, in the thermodynamical limit, the critical range for synchronization is equal to the lattice dimension. Many of our results could be extended to lattices of continuous-time oscillators, and hence have an even wider range of applicability.

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